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PERIODIC MOTIONS OF A GYROSTAT WITH A FIXED POINT IN A NEWTONIAN FIELD

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Poincaré's method of small parameter is used to investigate the existence of periodic motions of a gyrostat with a fixed point, consisting of a rigid body and a rotor. The friction at the rotor shaft and other dissipative effects are neglected. The gyrostatic moment is assumed constant. The free Euler — Poinsot motion is chosen to represent the generating solution.

The motion of the gyrostat above a fixed point in a central Newtonian gravity field is described by a Hamiltonian which in the canonical Andoyer variables L_1, G_1, H_1, l_1, g_1, h_1 [1, 2] has the form

$$K = \frac{G_1^2 - L_1^2}{2AB}(A\cos^2 l_1 + B\sin^2 l_1) + \frac{L_1^2}{2C} - \tag{1}$$

$$\sqrt{G_1^2 - L_1^2}\left(\frac{k_1}{A}\sin l_1 + \frac{k_2}{B}\cos l_1\right) - \frac{k_3}{C}L_1 - U$$

$$U = -P(x_c\gamma_1 + y_c\gamma_2 + z_c\gamma_3) - \frac{3P}{2mR}(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$$

Here A, B and C are the principal moments of intertia of the gyrostat relative to the fixed point, k_1, k_2 and k_3 are the projections of the gyrostatic moment on the principal axes of inertia, P denotes the weight and m the mass of the gyrostat x_c, y_c and z_c are the coordinates of the center of mass of the gyrostat in the body coordinate system and γ_1, γ_2 and γ_3 are the direction cosines of the radius vector R of the fixed point of the gyrostat emerging from the center of attraction, in the body coordinate system.

Let us pass to the canonical action-angle variables L, G, H, l, g, h. The expansions of the Andoyer variables L_1, G_1, H_1, l_1, g_1, h_1 is terms of the action-angle variables L, G, H, l, g, h were given in [3]. The Hamiltonian of the problem is given in terms of these variables by the expression

$$K = \frac{\bar{L}^2}{2D} + \frac{1}{4}\left(\frac{1}{A} + \frac{1}{B}\right)G^2 + \tag{2}$$

$$\sum_{s_1, s_2}\left[a_{s_1, s_2}\left(\frac{L}{G}, \frac{H}{G}\right)\sin(s_1 l + s_2 g) + b_{s_1, s_2}\left(\frac{L}{G}, \frac{H}{G}\right)\times\right.$$

$$\left.\cos(s_1 l + s_2 g)\right] + \sum_{i_1, i_2} b_{i_1, i_2}\left(\frac{L}{G}, \frac{H}{G}\right)\cos(i_1 l + i_2 g)$$

$(s_1 = 1, 3, 5, \ldots;\ s_2 = -1, 0, +1;\ i_1 = 0, 2, 4, \ldots;$
$i_2 = -2, -1, 0, +1, +2)$

$$\bar{L} = L\left[1 + \frac{e^2}{16}(b-1)(b+3) + \dots\right], \quad b = \frac{G^2}{L^2}$$

$$\frac{1}{D} = \frac{1}{C} - \frac{1}{2}\left(\frac{1}{A} + \frac{1}{B}\right), \quad e = \frac{1}{2}\left(\frac{1}{B} - \frac{1}{A}\right) D$$

$$a_{s_1, \mp 1} = \frac{PL\sqrt{G^2 - H^2}\, x_c}{2G^2}\left(\mu_{s_1, \mp 1} + \frac{G}{L}\theta_{s_1, \mp 1}\right)$$

$$a_{s_1, 0} = \left(\frac{PHx_c}{G^2} - \frac{k_1}{A}\right)\sqrt{G^2 - L^2}\, \nu_{s_1, 0}$$

$$b_{s_1, \mp 1} = \frac{PL\sqrt{G^2 - H^2}\, y_c}{2G^2}\left(\varepsilon_{s_1, \mp 1} + \frac{G}{L}\varphi_{s_1, \mp 1}\right)$$

$$b_{s_1, 0} = \left(\frac{PHy_c}{G^2} - \frac{k_2}{B}\right)\sqrt{G^2 - L^2}\, \varkappa_{s_1, 0}$$

$$b_{i_1, \mp 1} = \frac{\sqrt{(G^2 - L^2)(G^2 - H^2)}}{G^4}\left\{\frac{3P(B-A)}{2mR}\times \right.$$

$$\left.\left[HL(1 - 2\sigma)\, d^{\circ}_{i_1, \mp 1} \mp \frac{H(G \mp L)}{2} d^2_{i_1, \mp 1}\right] + PG^2 z_c \psi_{i_1, \mp 1}\right\}$$

$$b_{i_1, 0} = \left(\frac{PHz_c}{G^2} - \frac{k_3}{C}\right) L\alpha_{i_1, 0} +$$

$$\frac{3P(B-A)}{8mRG^4}\{G^2(G^2 - H^2)(2\delta - 1)\,\delta_{i_1, 0} +$$

$$[(2\delta - 1) L^2 d^{\circ}_{i_1, 0} + (G^2 - L^2)\, d^2_{i_1, 0}]\}, \quad \delta = \frac{C - A}{B - A}$$

Here $\delta_{i_1, 0}$ is the Kronecker delta and the quantities $d^{n}_{i_1, s_2}, \mu_{s_1, \mp 1}, \theta_{s_1, \mp 1}, \nu_{s_1, 0}, \varepsilon_{s_1, \mp 1}$ $\varphi_{s_1, \mp 1}, \varkappa_{s_1, 0}$ $\psi_{i_1, \mp 1}, \alpha_{i_1, 0}$ are given by well known series written in terms of increasing powers of the parameter e, the coefficients of which depend on b. Since the coefficients $b_{i_1, \pm 2}$ are independent of the gyrostatic moment and of the coordinates of the mass center of the gyrostat, they coincide with the coefficients $U_{2m, \pm 2}$ appearing in the expansion describing the force function in [4]. The motion of the gyrostat is described by the canonical equations, and the system of equations admist two first integrals

$$K = C_1, \; H = C_2$$

We assume that the ellipsoid of inertia for the fixed point of the gyrostat differs little from the ellipsoid of revolution, i. e. the difference $A - B$ is small. Moreover, the center of mass of the gyrostat is situated sufficiently near the fixed point, and the gyrostatic moment is small. In this case the Hamiltonian function can be separated into two parts in such a manner, that the function μK_1 remains very small in modulo, within the domain of variation of the canonical variables, compared with the first term K_0

$$K = K_0(L, G) + \mu K_1\left(\frac{L}{G}, \frac{H}{G}, l, g\right)$$

$$K_0(L, G) = \frac{\bar{L}^2}{2D} + \frac{1}{4}\left(\frac{1}{A} + \frac{1}{B}\right) G^2$$

Here K_0 denotes the unperturbed Hamiltonian describing the Eulerian motion μK_1 is the perturbing Hamiltonian and μ is a small parameter. The Hamiltonian K_0 depends on the action variables L and G only, and the generating solution defined by it has the form

$$l_0 = n_1^{(0)}t + \omega_1, \quad L_0 = a_1$$
$$g_0 = n_2^{(0)}t + \omega_2, \quad G_0 = a_2$$
$$h_0 = \omega_3, \quad H_0 = a_3$$

where ω_i and a_i are arbitrary constants. The frequencies of the angle variables are equal to

$$n_1^{(0)} = \frac{a_1}{D}\left[1 - (b_0^2 + 3)\frac{e^2}{8} + \dots\right]$$
$$n_2^{(0)} = \frac{a_2}{2}\left(\frac{1}{A} + \frac{1}{B}\right) + \frac{a_2}{4D}(b_0 + 1)e^2 + \dots, \quad b_0 = \frac{a_2^2}{a_1^2}$$

When the frequencies $n_1^{(0)}$ and $n_2^{(0)}$ are commensurable, the motion becomes periodic.

Let us now consider the conditions of existence of periodic solutions of the system of equations with the Hamiltonian (2), which coincide with the generating solution when $\mu = 0$. The equations of perturbed motion admit, according to the results obtained by Poincaré [5], periodic solutions at small values of μ, provided that the generating solutions satisfy the conditions

$$\Delta_1(K_0) \neq 0, \quad \frac{\partial[K_1]}{\partial\omega_2} = 0, \frac{\partial[K_1]}{\partial a_3} = 0 \tag{3}$$
$$\Delta_2([K_1]) \neq 0, \quad [K_1] = \frac{1}{T}\int_0^T K_1 dt$$

Here Δ_1 is the Hessian of the unperturbed Hamiltonian in L_0 and G_0, and Δ_2 is the Hessian of the mean value of the perturbing Hamiltonian $[K_1]$ in a_3 and ω_2.

The first condition of (3) always holds, except in the case of complete dynamic symmetry. This is because we have, with the accuracy of up to e^2,

$$\Delta_1(K_0) = \frac{1}{2D}\left(\frac{1}{A} + \frac{1}{B}\right) + \frac{c^2}{16D^2} \times$$
$$\left[4(3b_0 + 1) + 3D\left(\frac{1}{A} + \frac{1}{B}\right)(b_0^2 - 1) + \dots\right] \neq 0$$

To investigate the remaining conditions of periodicity, we must compute the mean value of the perturbing function K_1 over a single period. The following cases are possible:

$$1)\ (2N-1)\,n_1^{(0)} = n_2^{(0)}, \quad 2)\ 2Nn_1^{(0)} = n_2^{(0)}$$

and we obtain for the function $[K_1]$, respectively,

$$1)\ [K_1] = b_{0,0}\left(\frac{L_0}{G_0}, \frac{H_0}{G_0}, k_3, z_c\right) + a_{2N-1,-1}\sin\eta +$$
$$b_{2N-1,-1}\cos\eta + b_{2(2N-1),-2}\cos 2\eta, \quad \eta = (2N-1)\,l_0 - g_0$$

$$2)\ [K_1] = b_{0,0}\left(\frac{L_0}{G_0}, \frac{H_0}{G_0}, k_3, z_c\right) + b_{2N,-1}\cos\xi + b_{4N,-2}\cos 2\xi$$
$$\xi = 2Nl_0 - g_0$$

where N is a positive integer. The coefficients $a_{2N-1,-1}, b_{0,0}, b_{2N-1,-1}, b_{2(2N-1),-2}$, $b_{2N,-1}$, $b_{4N,-2}$ can be found from (2), with the action variables L_0, G_0 and H_0 assuming the generating values.

The second condition of (3) yields the relation for the generating values of the angle variables l, g and h. In the case of commensurability 1) this relation assumes the form

$$a_{2N-1,\,-1}\cos\eta - b_{2N-1,\,-1}\sin\eta - 2b_{2\,(2N-1),\,-2}\sin 2\eta = 0$$

In the case of commensurability 2), the values of l_0, g_0 and h_0 coincide with the corresponding values obtained in [4]. The third condition of (3) holds in the case of commensurability 1) when $z_c = 0$ and $H_0 = 0$, with x_c and y_c arbitrary. The last condition of (3) always holds.

Thus we have shown that the problem of motion of a gyrostat in a central Newtonian force field admits a family of periodic Poincaré solutions.

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